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EQUIMULTIPLE DEFORMATIONS OF ISOLATED SINGULARITIES

BY

I. SCHERBACK AND E. SHUSTIN*

School of Mathematical Sciences, Tel Aviv University
Ramat Aviv, Tel Aviv 69978, Israel
e-mail: scherbak@post.tau.ac.il, shustin@post.tau.ac.il

ABSTRACT

We study equimultiple deformations of isolated hypersurface singularities, introduce a blow-up equivalence of singular points, which is intermediate between topological and analytic ones, and give numerical sufficient conditions for the blow-up versality of the equimultiple deformation of a singularity or multisingularity induced by the space of algebraic hypersurfaces of a given degree. For singular points, which become Newton nondegenerate after one blowing up, we prove that the space of algebraic hypersurfaces of a given degree induces all the equimultiple deformations (up to the blow-up equivalence) which are stable with respect to removing monomials lying above the Newton diagrams. This is a generalization of a theorem by B. Chevallier.

Introduction

This paper is devoted to deformations of a special kind for isolated hypersurface singularities over the real or complex field.

A classical problem is to describe what happens with singularities of an algebraic hypersurface when varying in the space of hypersurfaces of a given degree (or in another interesting class). The versality of a deformation guarantees that it contains all possible deformations in the considered class. The case of algebraic

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curves has been studied intensively; we refer the reader to the survey [13] for an up-to-date account and bibliography. Few results have been obtained for higher dimensions [6, 7, 8, 22]. However, the ordinary versality (i.e., with respect to the group of all diffeomorphisms) appears to be a rather restrictive condition, which does not hold for a wide range of the hypersurface spaces (see examples in [8, 11]).

We approach this main problem from two sides. First, we study versality of deformations with respect to different equivalences, i.e., defined by groups of diffeomorphisms or homeomorphisms of certain types. We consider **equimultiple deformations**, i.e., deformations in the class of the germs of a given multiplicity. For some class of singular points (called **NDT — non-degenerate along tangents**), we define **blow-up equivalence**, which is a topological equivalence in the original space and local analytic equivalence in the blown-up space (section 1.1.5). We study equimultiple deformations of NDT singular points, which are versal with respect to the blow-up equivalence. Such versality is stronger than versality up to topological equivalence, but, on the other hand, it can be induced by the spaces of hypersurfaces of bounded degrees (see Theorems 1, 2, 3 and Corollary 2). We show that equimultiple blow-up versal deformations of NDT singularity or multisingularity (i.e., a collection of singularities) are realizable in the space of algebraic hypersurfaces satisfying certain numerical conditions.

Another point of view on the versality problem arises from the Viro method [24, 25, 26, 18] and its modifications [20, 21] which produce certain one-parametric deformations of Newton nondegenerate singular points ("lower deformations" in terms of [21]). B. Chevallier [4] introduced NNDT singularities (Newton nondegenerate along tangents), which generalize Newton nondegenerate ones, and defined similar one-parametric deformations. An important advantage of these constructions is that they do not increase the degree of a given algebraic hypersurfaces. Another useful property is that the hypersurfaces in the Viro-type deformations admit an explicit topological description, namely, they are topologically glued out of pieces of sample algebraic hypersurfaces. Keeping these advantages, we extend the range of realizable deformations for NNDT singular points so that the whole family of these deformation possesses the following versality property: it contains all the (equimultiple) deformations which are stable, in a sense, with respect to removing monomials lying above the Newton diagrams (section 2.2). The meaning of stability is specified in the form of $\mathcal{S}$-transversality (see section 2.3.3), expressed in terms of the geometry of equisingular families of hypersurfaces. Theorem 4 assures that all $\mathcal{S}$-transversal deformations are induced by the space of hypersurfaces of the given degree, generalizing thereby Chevallier's theorem [4].

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1. Deformations of isolated singular points

1.1 BASIC DEFINITIONS AND NOTATIONS. Throughout the paper we always assume that the objects under consideration (functions, hypersurfaces, diffeomorphisms) are defined over $\mathbb{C}$, and the real case means that all the data are equivariant, i.e., invariant with respect to the complex conjugation. Moreover, all the statements formulated for the complex case are valid for the real case as well if one appends the equivariance condition.

1.1.1 Versal deformations. Versality of a deformation can be defined in various ways. We shall use the following definition.

Let M be a topological space. A deformation of an element $f \in M$ is a continuous map $F : (\Lambda, 0) \to (M, f)$, where Λ is a finite-dimensional linear $\mathbb{C}$-space, the base of the deformation. Let $M = \bigcup_{\alpha \in \mathcal{A}} M_\alpha$ be a subdivision into disjoint subsets M_α, $\alpha \in \mathcal{A}$. A deformation F of $f \in M$ is called **versal with respect to the given subdivision of** M, if there exists a neighborhood U of f in M such that the image of F intersects any subset M_α with $M_\alpha \cap U \neq \emptyset$.

A classical example is the following. Let M be an analytic manifold over $\mathbb{C}$ and G Lie group acting on M. A **deformation** of an element $f \in M$ is an analytic embedding germ $F : (\Lambda, 0) \to (M, f)$, where Λ is a finite-dimensional linear $\mathbb{C}$-space. The manifold M is decomposed into the orbits of the group action, and a deformation F of f is called **versal** (**G-versal**) if the image of F intersects all orbits in a neighborhood of f.

It is well-known [17] that if the G-orbit Gf of the element $f \in M$ is smooth then a deformation F: $(\mathbb{C}^s, 0) \to (M, f)$ is versal, provided

$$F_1 = \frac{\partial F}{\partial \lambda_1}\Big|_{\lambda=0}, \ldots, F_s = \frac{\partial F}{\partial \lambda_s}\Big|_{\lambda=0}$$

span in $T_f M$ a subspace $T_f F$ transversal to the tangent space $T(Gf)_f$ to the orbit (here $\lambda_1, \ldots, \lambda_s$ are coordinates in $\Lambda = \mathbb{C}^s$).

Below we consider deformations of isolated hypersurface singular points versal with respect to some group actions and with respect to some other equivalence relations in the main space.

1.1.2 Singularities of hypersurfaces, ideals and zero-dimensional schemes. Let $f\colon (\mathbb{C}^n, x) \to (\mathbb{C}, 0)$ be an analytic function germ having an isolated critical point at x. The group $\mathrm{Diff}^0_x(\mathbb{C}^n)$ of analytic diffeomorphisms of $(\mathbb{C}^n, x)$, preserving the point x, acts on the algebra of the analytic function germs,

$$\mathcal{O}_{\mathbb{C}^n,x} = \{g\colon (\mathbb{C}^n, x) \to (\mathbb{C}, 0)\}.$$

It is well-known [16, 23] that the power $\mathfrak{m}_x^k$ of the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{\mathbb{C}^n,x}$, for a sufficiently large k, is contained in the orbit of f with respect to the action of $\mathrm{Diff}^0_x(\mathbb{C}^n)$. Thus, the study of versal deformations with respect to the action of G being $\mathrm{Diff}^0_x(\mathbb{C}^n)$ or a larger group, is reduced to the case of a finite-dimensional manifold $\mathcal{O}_{\mathbb{C}^n,x}/\mathfrak{m}_x^k$ and the Lie group $G/\mathfrak{m}_x^k$.

Besides the group $\mathrm{Diff}^0_x(\mathbb{C}^n)$ defining an equivalence of function germs at the fixed critical point x, we shall consider the groups:

- $\mathrm{Diff}_x(\mathbb{C}^n) = \mathrm{Tran}(\mathbb{C}^n) \ltimes \mathrm{Diff}^0_x(\mathbb{C}^n)$, the semidirect product with a germ of the group of translations in $\mathbb{C}^n$, acting as $(T\Phi)_* g = g \circ T \circ \Phi$, $T \in \mathrm{Tran}(\mathbb{C}^n)$, $\Phi \in \mathrm{Diff}^0_x(\mathbb{C}^n)$, $g \in M$, and defining an equivalence of singularities of functions in a neighborhood of $f \in M$;
- $\mathbb{C}^* \times \mathrm{Diff}^0_x(\mathbb{C}^n)$ acting as $(a\Phi)_* g = ag \circ \Phi$ and defining an equivalence of hypersurface germs with the fixed singular point x;
- $\mathbb{C}^* \times \mathrm{Diff}_x(\mathbb{C}^n)$ defining an equivalence of hypersurface germs in a neighborhood of x in $\mathbb{C}^n$.

Germs of the orbits of f in $\mathcal{O}_{\mathbb{C}^n,x}/\mathfrak{m}_x^k$ under the action of the groups $\mathbb{C}^* \times \mathrm{Diff}_x(\mathbb{C}^n)$, $\mathbb{C}^* \times \mathrm{Diff}^0_x(\mathbb{C}^n)$ are smooth, and their tangent spaces are the ideals

$$I(f) = \left\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle, \quad I^0(f) = \langle f \rangle + \mathfrak{m}_x \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle,$$

respectively. These ideals define zero-dimensional schemes at x ([5]) which we denote as $X_x^{ea}(f)$, $X_x^{ea,0}(f)$, respectively. The scheme degrees are $\deg X_x^{ea}(f) = \tau(f)$, $\deg X_x^{ea,0}(f) = \tau(f) + n$, where $\tau(f)$ is the Tjurina number.

1.1.3 Equimultiple versal deformations. In the above notation, we define the multiplicity of the point x for the function f, or for the hypersurface $\{f = 0\}$ as

$$m = \mathrm{mult}_x(f) = \max\{l : f \in \mathfrak{m}_x^l\},$$

or, equivalently,

$$\mathrm{jet}_x^{m-1}(f) = 0, \ \mathrm{jet}_x^m(f) \neq 0.$$

Clearly, the multiplicity is an analytic invariant. The set of functions close to f with multiplicity m at x form a germ at f of the ideal $\mathfrak{m}_x^m$; the set of functions close to f with a singular point in a neighborhood of x having multiplicity m form a germ $\mathcal{M}_x^m$ of a smooth subvariety of $\mathcal{O}_{\mathbb{C}^n,x}/\mathfrak{m}^k$.

A deformation of f inside $\mathfrak{m}_x^m$ (resp., $\mathcal{M}_x^m$) is called **fixed-equimultiple** (resp., **equimultiple**). The group $\mathbb{C}^* \times \mathrm{Diff}_x(\mathbb{C}^n)$ acts in $\mathcal{M}_x^m$, and the group $\mathbb{C}^* \times \mathrm{Diff}_x^0(\mathbb{C}^n)$ acts in $\mathfrak{m}^m$, and we respectively define versal deformation:

- a finite-parametric deformation of the singular point x of the hypersurface germ $\{f = 0\}$ inside $\mathcal{M}_x^m$ is called **versal equimultiple**, if it crosses the $\mathbb{C}^* \times \mathrm{Diff}_x(\mathbb{C}^n)$-orbits of all germs in $\mathcal{M}_x^m$ sufficiently close to $\{f = 0\}$,
- a finite-parametric deformation of the singular point x of the hypersurface germ $\{f = 0\}$ inside $\mathfrak{m}_x^m$ is called **versal fixed-equimultiple**, if it crosses the $\mathbb{C}^* \times \mathrm{Diff}_x^0(\mathbb{C}^n)$-orbits of all germs in $\mathfrak{m}_x^m$ sufficiently close to $\{f = 0\}$.

As mentioned in section 1.1.1, according to [17] the following (infinitesimal) criteria suffice for the versality of deformations:

- an embedding $F\colon (\Lambda, 0) \to (\mathcal{M}_x^m, f)$ satisfying the condition that $T_f F$ is transverse to $I(f)$ in $T_f(\mathcal{M}_x^m)$, is a versal equimultiple deformation of the singular point x of the hypersurface germ $\{f = 0\}$;
- an embedding $F\colon (\Lambda, 0) \to (\mathfrak{m}_x^m, f)$ satisfying the condition that $T_f F$ is transverse to $I^0(f)$ in $\mathfrak{m}_x^m$, is a versal fixed-equimultiple deformation of the singular point x of the hypersurface germ $\{f = 0\}$.

In particular, one can obtain a versal fixed-equimultiple deformation of the germ f in the form $F(\lambda) = f + \sum_{i=1}^{s} \lambda_i F_i$, where $F_1, \ldots, F_s$ are the monomials of degree $\geq m$ of a monomial basis of $\mathcal{O}_{\mathbb{C}^n,x}/I^0(f)$.

1.1.4 Boundary singularities. We recall some definitions and facts concerning boundary singularities (see [19]).

A pair (g, H) consisting of an analytic function germ g and a smooth hypersurface germ H at the same point $x \in \mathbb{C}^n$ is called an H-boundary function germ. Correspondingly a pair (G, H), where $G = \{g = 0\}$, is called an H-boundary hypersurface germ. The point x is an isolated H-boundary singular point for (G, H) if x is an isolated critical point both for the function germ g and for the restriction to the boundary, $g|_H$. Equivalence of H-boundary singular points (in a neighborhood of x) is defined by the action of the group $\mathbb{C}^* \times \mathrm{Diff}_x(\mathbb{C}^n|H)$ in $\mathcal{O}_{\mathbb{C}^n,x}$, where $\mathrm{Diff}_x(\mathbb{C}^n|H) \subset \mathrm{Diff}_x(\mathbb{C}^n)$ is a subgroup, consisting of the diffeomorphism germs which take H to itself.

For an isolated H-boundary singular point x of (G, H), there exists k such that $\mathfrak{m}_x^k$ is contained in the orbit of g under the $\mathbb{C}^* \times \mathrm{Diff}_x(\mathbb{C}^n|H)$ action, hence the study of versal deformations of H-boundary singular point x of (G,H) is reduced to the finite-dimensional case of the manifold $\mathcal{O}_{\mathbb{C}^n,x}/\mathfrak{m}_x^k$ and the group $\mathbb{C}^* \times \mathrm{Diff}_x(\mathbb{C}^n|H)/\mathfrak{m}_x^k$ acting on it.

Without loss of generality assume that H is the hyperplane $x_n = 0$. Then the orbit of g under the $\mathbb{C}^* \times \mathrm{Diff}_x(\mathbb{C}^n|H)$-action is smooth and its tangent space at g is the ideal (see [19])

$$I(g|H) = \left\langle g, \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_{n-1}}, x_n\frac{\partial g}{\partial x_n}\right\rangle, \tag{1}$$

so, by [17], a deformation F: $(\Lambda, 0) \to (\mathcal{O}_{\mathbb{C}^n,x}, g)$ is a **versal deformation of the H-boundary singular point x of (G,H)** if T_fF is transverse to $I(g,H)$ in $\mathcal{O}_{\mathbb{C}^n,x}$. Denote by $X_{x|H}^{ea}$ the zero-dimensional scheme in $\mathbb{C}^n$ concentrated at x and defined by the ideal $I(g|H)$.

1.1.5 NDT hypersurface germs and blow-up equivalence. Let $W = \{f = 0\}$ be a hypersurface germ at an isolated singular point $x \in \mathbb{C}^n$. Consider the blowing-up π: $\Sigma \to \mathbb{C}P^n \supset \mathbb{C}^n$ of the point x. Denote by W^* the proper transform of W, by E the exceptional divisor, $E = \pi^{-1}(x)$. The germ W is called **non-degenerate along tangents** (briefly, **NDT**), if the set $\mathrm{Sing}(W^*|E) = \mathrm{Sing}(W^*) \cup \mathrm{Sing}(W^* \cap E)$ is finite and is contained in $W^* \cap E$. This implies, in particular, that W^* meets E transversally at any point $z \in W^* \cap E \setminus \mathrm{Sing}(W^*|E)$. We notice that any point $z \in \mathrm{Sing}(W^*|E)$ corresponds to a certain straight line $L_z \subset \mathbb{C}^n$ through x, tangent to W.

For example, any isolated singular point of a planar curve is NDT. Note also that any element $f' \in \mathfrak{m}_x^m$, which is sufficiently close to f, has an NDT singularity at x.

Clearly, the NDT-property of a singular point is invariant with respect to the $\mathbb{C}^* \times \mathrm{Diff}_x^0(\mathbb{C}^n)$-action. An element $\varphi \in \mathbb{C}^* \times \mathrm{Diff}_x^0(\mathbb{C}^n)$ defines a diffeomorphism φ^* of a neighborhood of E in Σ, which satisfies $\varphi^*\big|_E \in \mathrm{Aut}(E)$ and transforms a point $x^* \in \mathrm{Sing}(W)$ into an equivalent E-boundary singular point.

We shall define the **blow-up equivalence** of NDT singular points in the following way. Two NDT singular points x and x' of hypersurface germs $W, W' \subset \mathbb{C}^n$ are **blow-up equivalent**, $(W,x) \sim (W',x')$, if, in the above notation, there exists a C^∞-diffeomorphism of a neighborhood of the exceptional divisor E to a neighborhood of the exceptional divisor E', which takes E onto E', establishes one-to-one correspondence between $\mathrm{Sing}(W^*|E)$ and $\mathrm{Sing}(W'^*|E')$, and is analytic in a neighborhood of $\mathrm{Sing}(W^*|E)$ in Σ (thus, defines an analytic equivalence between E-boundary singular points in $\mathrm{Sing}(W^*|E)$ and E'-boundary singular points in $\mathrm{Sing}(W'^*|E')$, respectively).

The blow-up equivalence of NDT singularities lies between the analytic and topological equivalence: the diffeomorphism φ drops down to $\mathbb{C}^n$ as a bi-Lipschitz homeomorphism of neighborhoods of x and x'. For example, ordinary singular points of the same multiplicity $m = \text{mult}_x(f) = \text{mult}_{x'}(f')$, such that the lower homogeneous form of f and f' at x and x', respectively, has no critical points in $\mathbb{C}^n \setminus \{0\}$, are both topologically and blow-up equivalent, but not analytically if $n = 2$, $m \geq 4$, or $n \geq 3$, $m \geq 3$.

Example 1: Let $n \geq 3$, and $f(x_1, \ldots, x_n) = f_m(x_1, \ldots, x_n) + f_{m+1}(x_1, \ldots, x_n)$, where f_m and f_{m+1} are homogeneous polynomials of degrees m and $m+1$, respectively. The equation $f_m = 0$ defines a hypersurface in the projective space $\mathbb{C}P^{n-1}$. Assume that the hypersurface $\{f_m = 0\} \subset \mathbb{C}P^{n-1}$ has only isolated singular points, which we denote as $z_1, \ldots, z_r$. Suppose also that f_{m+1} is generic. Under these assumptions, the hypersurface $W = \{f = 0\}$ has an NDT singularity at the origin. Indeed, the blow-up W^* of W is nonsingular, and $\text{Sing}(W^*|E) = \{z_1, \ldots, z_r\}$ is just the non-transverse intersection locus of W^* and E. The germ at f of the blow-up equivalence stratum can be described as

$$\widetilde{f}_m(x_1, \ldots, x_n) + f_{m+1}(x_1, \ldots, x_n) + \widetilde{\mathfrak{m}}_x^{m+1},$$

where a m-form $\widetilde{f}_m$ defines in $\mathbb{C}P^{n-1}$ a hypersurface in a germ at $\{f_k = 0\}$ of the equisingular stratum in the space of hypersurfaces of degree m, and $\widetilde{\mathfrak{m}}_x^{m+1}$ is a germ of the linear space $\mathfrak{m}_x^{m+1}$.

1.1.6 Blow-up versal deformations. In the above notation, we define a **(fixed) blow-up versal deformation** (briefly, **(fixed) BUV deformation**) of an NDT singular point x of a hypersurface W as a (fixed) equimultiple deformation, which meets the blow-up equivalence classes for all $f' \in \mathcal{M}_x^m$ (respectively, $f' \in \mathfrak{m}_x^m$),* which are sufficiently close to f.

This versality can be described via blowing-up $\pi\colon \Sigma \to \mathbb{C}^n$ of the point x. Any $g \in \mathfrak{m}_x^m$ is lifted upon Σ as a function $g \circ \pi$ in a neighborhood of E. For any $x^* \in \text{Sing}(W^*|E)$ the germ of $g \circ \pi$ at x^* is divisible by the germ $E_{x^*}^m \in \mathcal{O}_{\Sigma,x^*}$, where $E_{x^*} = 0$ is an irreducible equation of E in a neighborhood of x^*. Put $g_{x^*} = (g \circ \pi)/E_{x^*}^m$. Then a fixed-equimultiple deformation $F\colon (\Lambda, 0) \to (\mathfrak{m}_x^m, f)$ is a fixed BUV deformation, if, for any point $x^* \in \text{Sing}(W^*|E)$, the family $F_{x^*}\colon (\Lambda, 0) \to \mathcal{O}_{\Sigma,x^*}$ is a versal deformation of the E-boundary singular point x^* of W^*.

* Here $\mathcal{M}_x^m$ is defined as in 1.1.3.

Since the analytic equivalence refines the blow-up equivalence, any versal (fixed) equimultiple deformation of an NDT singular point is (fixed) BUV as well.

Remark 1: We should point out a difference between the (fixed) blow-up versality and (fixed) equimultiple versality. Namely, the (fixed) blow-up versality in general does not reduce to an infinitesimal versality in the sense of [17], i.e., to the transversality of the intersection of a deformation and the equivalence stratum. The reason is that the blow-up equivalence stratum is not necessarily smooth, which we demonstrate in an example below. Next we provide a sufficient condition for the smoothness of the blow-up equivalence stratum and for the blow-up versality of an equimultiple deformation.

In the above notation, define ideals

$$I^{ndt,0}(f) = \{g \in \mathfrak{m}_x^m \colon g_{x^*} \in I(f_{x^*}|E) \text{ for all } x^* \in \mathrm{Sing}(W^*|E)\},$$
$$I^{ndt}(f) = I^{ndt,0}(f) + \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle.$$

The ideals $I^{ndt,0}(f)$, $I^{ndt}(f)$ define zero-dimensional schemes $X_x^{ndt,0}, X_x^{ndt} \subset \mathbb{C}^n$, respectively. In fact, $X_x^{ndt,0}$ is the blow-down of the union of the schemes $X^{ea}_{x^*|E}$, $x^* \in \mathrm{Sing}(W^*|E)$, which we denote as $\widetilde{X}_x^{ndt,0}$.

PROPOSITION 1: (1) *Let $\widetilde{L} = \pi^{-1}(L) \subset \Sigma$ be the preimage of a hyperplane $L \subset \mathbb{C}P^n$, which does not pass through x. If, for some $k \geq m$,*

$$h^1(\Sigma, \mathcal{J}_{\widetilde{X}_x^{ndt,0}/\Sigma} \otimes \mathcal{O}_\Sigma(k\widetilde{L} - mE)) = 0, \tag{2}$$

then the germ at f of the blow-up equivalence stratum in $\mathfrak{m}_x^m$ is smooth and has codimension

$$\deg \widetilde{X}_x^{ndt,0} = \deg X_x^{ndt,0} - \binom{m+n-1}{n}. \tag{3}$$

(2) *Under condition (2), a deformation $F\colon (\Lambda, 0) \to (\mathfrak{m}_x^m, f)$ is fixed BUV if T_fF is transverse to $I^{ndt,0}(f)$ in $\mathfrak{m}_x^m$. Similarly, a deformation $F\colon (\Lambda, 0) \to (\mathcal{M}_x^m, f)$ is BUV if T_fF is transverse to $I^{ndt}(f)$ in $T_f(\mathcal{M}_x^m)$.*

Proof: (1) The singularity at x defined by f is analytically equivalent to that defined by the $(\mu+1)$-jet of f, where μ is the Milnor number (see [23]). Hence it is sufficient to verify (2) only for $k = \mu + 1$. Then the blow-up equivalence stratum in $\mathfrak{m}_x^m$ can be reduced mod $\mathfrak{m}_x^{\mu+2}$. The blowing-up $\pi\colon \Sigma \to \mathbb{C}P^n$ takes the blow-up equivalence stratum in $\mathfrak{m}_x^m/\mathfrak{m}_x^{\mu+2}$ into the family in the linear system $|\mathcal{O}_\Sigma(k\widetilde{L}-mE)|$, consisting of hypersurfaces with E-boundary singularities analytically equivalent to these of W^*. By the basic deformation theory (see, for example, [9]), relation (2) with $k=\mu+1$ suffices for the smoothness and the expected codimension (3) of the latter E-boundary equisingular stratum.

(2) The condition of the transversal intersection of T_fF and $I^{ndt,0}(f)$ in $\mathfrak{m}_x^m$ lifts by π to the transversality of the intersection of the spaces $T_{f^*}F^*$ and $H^0(\Sigma, \mathcal{J}_{\widetilde{X}_x^{ndt,0}/\Sigma}\otimes\mathcal{O}_\Sigma(k\widetilde{L}-mE))$ in $H^0(\Sigma,\mathcal{O}_\Sigma(k\widetilde{L}-mE))$ for some k, which is the infinitesimal sufficient criterion for the joint versality of the deformation F^* with respect to all E-boundary singular points $X^*\in\mathrm{Sing}(W^*|E)$.

The second transversality condition reduces to the first one. ∎

Coming back to Example 1, we can derive that the smoothness of the blow-up equivalence stratum at f reduces to the smoothness of the equisingular stratum at $\{f_m=0\}$ in the space of hypersurfaces of degree m in $\mathbb{C}P^{n-1}$. Various sufficient criteria for the smoothness of equisingular families of hypersurfaces of a given degree and examples of non-smooth equisingular families can be found in [11, 12, 13, 7, 8, 21, 22]. In this case, the blow-up versality of a fixed-equimultiple deformation F means that the induced deformation of m-forms F_m is a joint versal deformation of the singularities of the hypersurface $\{f_k=0\}$ in $\mathbb{C}P^{n-1}$.

1.2 Bases of the Local Ring. In a similar way we study critical points of holomorphic germs and singular points of hypersurfaces.

Consider a holomorphic germ $f:(\mathbb{C}^n,O)\to(\mathbb{C},0)$ and a hypersurface germ $W=\{f=0\}$, where $f(x)=\sum a_mx^m$ in some coordinates $x_1,\ldots,x_n$. The local ring of the critical point is $Q(f)=\mathcal{O}_{\mathbb{C}^n,0}/J(f)$, whereas the local ring of the singular point is $S(f)=\mathcal{O}_{\mathbb{C}^n,0}/I(f)$, where

$$J(f)=\left\langle\frac{\partial f}{\partial x_i},\ 1\le i\le n\right\rangle,\quad I(f)=\left\langle f,\ \frac{\partial f}{\partial x_i},\quad 1\le i\le n\right\rangle.$$

If f is a quasihomogeneous germ, then $S(f)=Q(f)$ ([1], Ch. II, sec. 12).

1.2.1 Special bases of the local ring. Here we prove the existence of a basis of the local ring satisfying certain restrictions on the degrees of its monomials.

Representatives $e_1,\ldots,e_\mu$ of a basis of the local ring give the versal deformation: $F(x,\lambda)=f(x)+\sum_{i=1}^{\mu}\lambda_ie_i$. We assume that the Newton diagram of f, Γ, intersects coordinate axes in $\mathbb{Z}_+^n$ at points $\{(0,\ldots,0,l_i,0,\ldots,0),\quad i=1,\ldots,n\}$. Consider the following parallelepiped in $\mathbb{Z}_+^n$:

$$\Pi=\{k\in\mathbb{Z}_+^n|\ 0\le k_i<(l_i-1),\ i=1,\ldots,n\}.$$

PROPOSITION 2: *If f is a Newton non-degenerate germ, then there exists a basis of the local ring $Q(f)$ such that all its monomials lie in Π*

Proof: If the function germ is of the form $a_1x_1^{l_1}+\cdots+a_nx_n^{l_n}$, then the Jacobian ideal is generated by monomials $x_1^{l_1-1},\ldots,x_n^{l_n-1}$, and all monomials from Π are in the monomial basis of the local ring.

Consider now the germ f_Γ. The Jacobian ideal $J(f_\Gamma)$ is generated by

$$\frac{\partial f_\Gamma}{\partial x_i} = a_il_ix_i^{l_i-1} + \text{ monomials from } \bar{\Pi}, \quad i=1,\ldots,n,$$

where $\bar{\Pi}=\{k\in\mathbb{Z}_+^n\colon 0\le k_i\le(l_i-1),\ i=1,\ldots,n\}$ is the closure of Π. Thus we get

$$x_i^{l_i-1} \equiv \text{ a sum of monomials from } \bar{\Pi} \mod J(f_\Gamma), \quad i=1,\ldots,n.$$

From the convexity of Γ, it follows that

$$x_i^{l_i-1} \equiv \text{ a sum of monomials from } \Pi \mod J(f_\Gamma), \quad i=1,\ldots,n. \tag{4}$$

Indeed, the worst case is when f_Γ is a homogeneous polynomial of the form (for simplicity, we write down a polynomial in two variables) $\alpha x^l+\beta x^{l-1}y+\gamma y^{l-1}x+\delta y^l+\cdots$, such that the coefficients $\alpha,\beta,\gamma,\delta$ satisfy the condition $\alpha\delta l^2=\gamma\beta$. In this case we cannot replace x^{l-1} by a sum of monomials from Π. But it is easy to see that almost all linear changes of variables and almost all perturbations of the coefficients destroy this condition.

Thus the set of all monomials from the right sides of the last congruences includes (or coincides with) all monomials from a basis of the local ring of f_Γ.

In the general case, we use the following result ([1], Ch. II. sec. 12):

If f is Newton non-degenerate, then it is smoothly equivalent to $f_\Gamma+\sum c_ie_i$, where $\{e_i\}$ are the monomials of a basis of $Q(f_\Gamma)$ which lie above Γ.

If we choose monomial representatives of a basis for $Q(f_\Gamma)$ which lie strictly in Π, then all monomials $\{e_i\}$ turn out to be in Π too, and hence for f the same congruences as in (2) hold. This completes the proof of the proposition. ∎

1.2.2 Generic bases of the local rings. Here we prove some restrictions on any basis of the local ring of a Newton non-degenerate singular point.

PROPOSITION 3: *If f is Newton non-degenerate germ, f_Γ contains monomials $x_i^{l_i}$, $i=1,\ldots,n$, and h is a minimal number such that $x_n^h\in I(f)$, then*

$$h\le l_n\left[n-2\left(\frac{1}{l_1}+\cdots+\frac{1}{l_n}\right)\right]+1.$$

Proof: In the "worst" case, when $f = f_\Gamma$ is a quasihomogeneous germ with weights $\{1/l_i,\ i = 1, \dots, n\}$, the maximal possible quasi-degree in a basis of the local ring is $\sum(l_i - 2)/l_i$. Indeed, the number of the monomials in a (monomial) basis of the local ring which have a given quasi-degree, is the same for any bases ([1], Ch. II, sec. 12), and for the germ $\sum x_i^{l_i}$, all monomials with quasi-degree bigger than $\sum(l_i - 2)/l_i$ lie in the Jacobian ideal (see the beginning of the proof of Proposition 1). The equation of the hyperplane through the points which correspond to the monomials $x_i^{l_i}$, $i = 1, \dots, n$ is

$$\frac{k_1}{l_1} + \cdots + \frac{k_n}{l_n} = 1;$$

here $k_1, \dots, k_n$ are coordinates in the lattice Z_+^n. The equation of the parallel hyperplane through the point $(l_1 - 2, \dots, l_n - 2)$ is

$$\frac{k_1}{l_1} + \cdots + \frac{k_n}{l_n} = \frac{l_1 - 2}{l_1} + \cdots + \frac{l_n - 2}{l_n}.$$

The hyperplane intersects axis k_n at the point

$$k_n = l_n\Big[n - 2\Big(\frac{1}{l_1} + \cdots + \frac{1}{l_n}\Big)\Big].$$

This is the maximal possible power of x_n (if it is integral) that can be not in $I(f)$. ∎

1.2.3 Boundary singularities case. Consider the E-boundary function germ (f, E). Choose local coordinates $x_1, \dots, x_n$, such that the boundary is $x_n = 0$. We denote by f_0 the restriction of f to the boundary:

$$f_0(x_1, \dots, x_{n-1}) = f(x_1, \dots, x_{n-1}, 0).$$

The germ (f, E) is an isolated boundary singularity if both the germs f and f_0 are isolated singularities (without boundary conditions). The Newton diagram $\Gamma(f_0)$ of f_0 is the intersection of $\Gamma(f)$ with the coordinate hyperplane $k_n = 0$ in $\mathbb{Z}_+^n$. If $\Gamma(f)$ has a point at any coordinate axis, then the same holds for $\Gamma(f_0)$. If f is a Newton non-degenerate germ, then f_0 is a Newton non-degenerate as well.

The local ring of the boundary germ (f, E) is ([19])

$$Q(f|E) = \mathcal{O}_{\mathbb{K}^n,0}/J(f,E), \quad J(f|E) = \Big\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{n-1}}, x_n\frac{\partial f}{\partial x_n} \Big\rangle.$$

For a germ of a hypersurface with boundary, the local ring is

$$S(f|E) = \mathcal{O}_{\mathbb{K}^n,0}/I(f|E), \quad I(f|E) = \Big\langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{n-1}}, x_n\frac{\partial f}{\partial x_n} \Big\rangle.$$

Note that if f is a quasi-homogeneous germ, then f_0 is a quasi-homogeneous as well (to get the Euler equality for f_0, it is enough to substitute $x_n = 0$ in the Euler equality for f and to notice that $\partial f/\partial x_i|_{x_n=0} = \partial f_0/\partial x_i$, $i \neq n$). Moreover, it is clear that in this case $I(f|E) = J(f|E)$.

There exists the following exact sequence of the local rings of the function germs f, f_0 and (f, E) [19]:

$$0 \to Q(f) \to Q(f, E) \to Q(f_0) \to 0;$$

here $Q(f) \to Q(f, E)$ is the multiplication by x_n, and $Q(f, E) \to Q(f_0)$ is the natural projection. In particular, if $\{e_i,\ 1 \le i \le \mu\}$ and $\{e_j^0,\ 1 \le j \le \mu_0\}$ are representatives of bases of local rings of germs f and f_0 respectively, then the set

$$\{x_n e_i,\ e_j^0,\ 1 \le i \le \mu,\ 1 \le j \le \mu_0\}$$

represents a basis of the local ring of the boundary singularity (f, E).

Under the above assumptions on germ f, we get the following result for the boundary germ (f, E).

COROLLARY 1: (i) *If (f, E) is a Newton non-degenerate boundary germ, then there exists a basis of the local ring such that all its monomials lie in the parallelepiped*

$$\Pi_0 = \{(k_1, \ldots, k_n) \colon 1 \le k_i \le l_i - 2,\ 1 \le i \le n-1,\ 1 \le k_n \le l_n - 1\}.$$

(ii) *If $x_n = 0$ is the boundary, then the minimal h such that $x_n^h \in I(f|E)$ satisfies*

$$h \le l_n\Big[n - 2\Big(\frac{1}{l_1} + \cdots + \frac{1}{l_n}\Big)\Big] + 2. \tag{5}$$

The beginning of the classification (up to the stable equivalence) of the boundary singularities consists of the following series of germs (the boundary $E = \{x = 0\}$):

$$B_k \colon f = x^k + y^2; \quad C_k \colon f = xy + y^k; \quad F_{2k} \colon f = y^k + x^2.$$

For these series in the case $n = 2$, the minimal h such that $x^h \in I(f|E)$ is

$$B_k: \quad h = k; \quad C_k: \quad h = 2; \quad F_{2k}: \quad h = 3.$$

2. BUV deformations of algebraic hypersurfaces of fixed degree

In this section we give numerical criteria for the spaces of hypersurfaces of a given degree to be BUV deformations for one or a few NDT singular points. In section 2.1 we give a general criterion for a simultaneous BUV deformation of several NDT points. In section 2.2 we give a general criterion for a BUV deformation of one **Newton nondegenerate along tangents** (briefly, NNDT) singular point. In the latter criterion the degree d' of the hypersurfaces realizing BUV deformation can be bigger than the degree d of the initial hypersurface. We show that $d' = d$ under some additional conditions; in section 2.2 this is done for a special class of NNDT singular points. In section 2.3 we show that $d' = d$ for an arbitrary NNDT singular point but for a restricted class of deformations, so-called lower deformations. In particular, Theorem 4 generalizes Theorem III.2 [4], where only the existence of non-singular lower equimultiple deformations is proven.

2.1 BUV Deformation of Several NDT Singular Points. Let $F \subset \mathbb{C}P^n$ be a hypersurface of degree d with NDT singular points $z_1, \ldots, z_m$ as its only singularities.

Theorem 1: *If*

$$\sum_{i=1}^{m} \deg X_{z_i}^{ndt} < 4d - 4,$$

then the germ at F of the linear system $|\mathcal{O}_{\mathbb{C}P^n}(d)|$ induces a joint BUV deformation of the singular points $z_1, \ldots, z_m$ of F.

Proof: Since $X_{z_i}^{ndt} \subset X_{z_i}^{ea}$, $1 \le i \le m$, the required statement can be derived in the same way as Theorems 1 and 2 [22] (see also [8]). ∎

In the case $n = 2$ we can say more.

Theorem 2: *In the previous notation, let $n = 2$. Then the germ at F of the linear system $|\mathcal{O}_{\mathbb{C}P^2}(d)|$ induces a joint BUV deformation of the singular points $z_1, \ldots, z_m$ of F, provided one of the following conditions is satisfied:*

(i) $\sum_{i=1}^{m}(\deg X_{z_i}^{ndt} - \operatorname{isod}_{z_i}(\mathcal{J}_{X_{z_i}^{ndt}/\mathbb{C}P^2}, \mathcal{O}_F)) < 4(d-1)$, *where* $\operatorname{isod}_z(\mathcal{J}_{X_{z_i}^{ndt}/\mathbb{C}P^2}, \mathcal{O}_F)$, *defined as in* [10], *section 3, always is a nonnegative integer, or*

(ii) *F is irreducible and*

$$\sum_{i=1}^{m} \gamma(F, X_{z_i}^{ndt}) \le (d+3)^2,$$

where $\gamma(F,Y)$ defined as in [12], section 2.1, satisfies $\gamma(F,Y) \le (\deg Y + 1)^2$ ([12], Lemma 2.2).

This follows from [12], Proposition 2.1, [10], Corollary 3.9.

2.2 BUV DEFORMATION OF ONE NNDT SINGULAR POINT. Consider a holomorphic function germ f at the origin of $\mathbb{C}^n$. In coordinates $x_1, \dots, x_n$ we write $f(x) = \sum a_m x^m$, where $m = (m_1, \dots, m_n)$ and $x^m = x_1^{m_1} \dots x_n^{m_n}$. Let Γ be the Newton diagram of f; denote by f_Γ the main part of f:

$$f_\Gamma = \sum_{m \in \Gamma} a_m x^m.$$

We say that f is **Newton non-degenerate**, if f_Γ has an isolated critical point at the origin, any coordinate axis in Z_+^n contains a point of Γ, and the truncations f^σ to the facets σ of Γ (i.e., the sums of monomials in f, corresponding to the integral points in σ) have no critical points in $(\mathbb{C}^*)^n$.

Let $F \subset \mathbb{C}P^n$ be an algebraic hypersurface. Denote by $\pi\colon \Sigma \to \mathbb{C}P^n$ the blowing-up of z. Let F^* be the proper transform of F, E be the exceptional divisor, $z_1, \dots, z_k$ be all the singular points of $F^* \cap E$. An NDT singular point $x \in F$ is called **Newton nondegenerate along tangents** (briefly, **NNDT**) if each point z_i, $1 \le i \le k$, is Newton non-degenerate with respect to some local coordinates $x_1, \dots, x_n$ such that $E = \{x_n = 0\}$, $L^*_{z_i} = \{x_1 = \dots = x_{n-1} = 0\}$.

Let z be an NNDT singular point of multiplicity m of an algebraic hypersurface $F \subset \mathbb{C}P^n$ of degree d. Let $X \subset \mathbb{C}P^n$ be a zero-dimensional scheme concentrated at z and defined by the m-th power of the maximal ideal $\mathfrak{m}_z^m \subset \mathcal{O}_{\mathbb{C}P^n, z}$, and let $\mathcal{J}_{X/\mathbb{C}P^n}$ be the ideal sheaf of X on $\mathbb{C}P^n$. Introduce the zero-dimensional schemes $X' = X^{ea}_{z_1}(F^* \cap E) \cup \dots \cup X^{ea}_{z_k}(F^* \cap E) \subset E$ and $Y = X^{ea}_{z_1|E} \cup \dots \cup X^{ea}_{z_k|E} \subset \Sigma$ defined as in sections 1.1.2 and 1.1.4. Denote $h(z) = \min\{k \in \mathbb{Z}\colon kE \supset Y\}$.

THEOREM 3: *In the above notation, put $d' = \max\{d,\ r + h(z) - 1\}$.*

(i) If $n = 2$, then the germ at F of the linear system $|\mathcal{J}_{X/\mathbb{C}P^n}(d')|$ induces a fixed BUV deformation of z.

(ii) If $n > 2$ and

$$H^1(E, \mathcal{J}_{X'/E}(r)) = 0, \tag{6}$$

then the germ at F of the linear system $|\mathcal{J}_{X/\mathbb{C}P^n}(d')|$ is a fixed BUV deformation of z.

Proof: First, we notice that the first part of Theorem 3 follows from the second one. Indeed, for $n = 2$, the scheme $X^{ea,0}_{z_j}(F^* \cap E)$, $1 \le j \le k$, on the line E is defined by the ideal $(\mathfrak{m}_{z_j})^{m_j-1} \subset \mathcal{O}_{E,z_j}$, where m_j is the intersection multiplicity of E and F^*, which immediately implies (6), since $m_1 + \cdots + m_k \le r$.

The fixed-BU-versality of the linear system $|\mathcal{J}_{X,\mathbb{C}P^n}(d')|$ means that the map $H^0(\mathbb{C}P^n, \mathcal{J}_{X/\mathbb{C}P^n}(d')) \to H^0(X_z^{ndt,0}, \mathcal{O}_{X_z^{ndt,0}})$ is surjective, which is equivalent to the surjectivity of the map $H^0(\Sigma, \mathcal{O}_\Sigma(d'\widetilde{L} - rE)) \to H^0(Y, \mathcal{O}_Y)$, where $\widetilde{L}$ is the strict transform of a generic hyperplane in $\mathbb{C}P^n$, and the latter surjectivity is equivalent to

$$(7) \qquad H^1(\Sigma, \mathcal{J}_{Y/\Sigma} \otimes \mathcal{O}_\Sigma(d'\widetilde{L} - rE)) = 0.$$

To establish (7), we apply the so-called "Horace's method" [14]. Introduce the zero-dimensional schemes

$$Y_0 = Y, \quad Z_i = Y_{i-1} \cap E, \quad Y_i = Y_{i-1} : E, \quad i = 1, \ldots, h(z),$$

where $Y_{i-1} : E$ denotes the residue scheme defined at each point z by the ideal $\{\varphi \in \mathcal{O}_{\Sigma,z}\colon E\varphi \in I\}$, I is the ideal of $(Y_{i-1})_z$. We have the following exact sequences:

$$\begin{aligned} 0 &\to \mathcal{J}_{Y_i/\Sigma} \otimes \mathcal{O}_\Sigma(d'\widetilde{L} - (r+i)E) \to \mathcal{J}_{Y_{i-1}/\Sigma} \otimes \mathcal{O}_\Sigma(d'\widetilde{L} - (r+i-1)E) \\ &\to \mathcal{J}_{Z_i/E} \otimes \mathcal{O}_E(r+i-1) \to 0, \quad i = 1, \ldots, h(z), \end{aligned}$$

that produce cohomology sequences, from which we can subsequently derive (7), provided

$$(8) \qquad H^1(E, \mathcal{J}_{Z_i/E} \otimes \mathcal{O}_E(r+i-1)) = 0, \quad i = 1, \ldots, h(z),$$

$$(9) \qquad H^1(\Sigma, \mathcal{J}_{Y_{h(z)}/\Sigma} \otimes \mathcal{O}_\Sigma(d'\widetilde{L} - (r+h(z))E)) = 0.$$

Note that $Y_{h(z)} = \emptyset$ by definition of $h(z)$, and $d' \ge r + h(z) - 1$ by definition of d', which immediately implies (9). On the other hand, $Z_1 = Y \cap E = X'$, $Z_1 \supset Z_2 \supset \cdots \supset Z_{h(z)}$, which implies (8) in virtue of (6). ∎

Estimates for $h(z)$ can be found in section 1.2.3 (see, for example, (5) for generic NNDT singular points). In particular, the computation in section 1.2.3 implies

COROLLARY 2: *The statement of Theorem 3 holds for $n = 2$ and $d' = d$, provided that all the boundary singular points $z_1, \ldots, z_k$ of F^* (with respect to E) are of types B_k, F_{2k}, or C_k.*

Example 2: Let $f(x_1,\dots,x_n) = f_m(x_1,\dots,x_n) + f_{m+r}(x_1,\dots,x_n)$, where f_m and f_{m+r} are nonzero homogeneous polynomials of degrees m and $m+r$, respectively. Assume that $n=2$, or $n=3$ and the hypersurface $\{f_m=0\}\subset \mathbb{C}P^{n-1}$ has only isolated singular points. Suppose also that f_{m+k} is generic. We claim that this singular point is NDT, and it satisfies the conditions of Theorem 3 with $d' = m+r$. Namely, in the notation of Theorem 3, it is sufficient to show that $h \le r$. Without loss of generality, suppose that $z_1 \in \mathrm{Sing}(\{f_k=0\}) \subset \mathbb{C}P^{n-1}$ has coordinates $(0,\dots,0,1)$. Representing the blow-up $\pi\colon Y \to \mathbb{C}P^n$ locally as the coordinate change

$$y_1 = x_1x_n, \quad \dots, \; y_{n-1} = x_{n-1}x_n, \; y_n = x_n,$$

we obtain an equation of the strict transform of the hypersurface $\{f=0\}$ in a neighborhood of z_1 in the form

$$g(y_1,\dots,y_n) = g_0(y_1,\dots,y_{n-1}) + y_n^r g_1(y_1,\dots,y_{n-1}) = 0, \quad g_1(0)\neq 0,$$

where $E=\{y_n=0\}$, and $g_0=0$ is an equation of the hypersurface $\{f_m=0\}\subset E$ in a neighborhood of z_1. By (1) the ideal of the scheme $X^{ea}_{z_1|E}$ contains an element

$$y_n\frac{\partial g}{\partial y_n} = ry_n^r g_1(y_1,\dots,y_{n-1}), \quad rg_1(0)\neq 0,$$

thus contains an element y_n^r, or, in other words, $E^r \supset X^{ea}_{z_1|E}$.

At last, we notice that, for $n\ge 3$, one should also verify condition (6), which means, in fact, that the space of hypersurfaces of degree m in $\mathbb{C}P^{n-1}$ induces a joint versal deformation of all singular points of the hypersurface $\{f_m=0\}\subset \mathbb{C}P^{n-1}$.

2.3 BUV Deformation of an NNDT Singular Point. We start with an adoption of the definition of a lower deformation from [21] to the case of NNDT singular points.

2.3.1 One-parametric deformations of singularities and their models. Let $F\colon (\mathbb{C}^n,O)\to(\mathbb{C},0)$ be an isolated boundary singular point with respect to a hyperplane $L=\{x_n=0\}$, and let B be a sufficiently small closed ball centered at $z=O\in\mathbb{C}^n$ such that $V_0=\{F=0\}\cap B$ and $W_0=V_0\cap L$ are compact varieties with an isolated singular point z and the boundaries $\partial V_0 = V_0\cap\partial B$, $\partial W_0 = \partial V_0 \cap L$, and, in addition, V_0 (resp., W_0) is transversal to ∂B along ∂V_0 (resp., ∂W_0) and is homeomorphic to a cone over ∂V_0 (resp., ∂W_0).

Under a one-parametric deformation of the boundary singular point $z\in V_0$ we mean an analytic hypersurface $\mathcal{V}\subset B\times D_\varepsilon$, $D_\varepsilon=\{t\in\mathbb{C}\colon |t|<\varepsilon\}$, such that

- $(\pi|_{\mathcal{V}})^{-1}(0) = V_0$, where $\pi: B \times D_\varepsilon \to D_\varepsilon$ is the projection,
- for $t \neq 0$, $(\pi|_{\mathcal{V}})^{-1}(t) := V_t$ and $W_t = V_t \cap L$ are compact varieties in B with isolated singularities in $\mathrm{Int}(B)$ and boundaries $\partial V_t \subset \partial B$, $\partial W_t \subset \partial B$, so that V_t, W_t are transversal to ∂B along ∂V_t, ∂W_t, respectively.

For any equivalence of singular points, there is the corresponding equivalence of boundary singular points defined by keeping the boundary (see, for instance, section 1.2.3). For both of them, we use the same notation. Given an equivalence $\mathcal{S}$ of hypersurface isolated singular points, usual and boundary, we say that a deformation V_t, $t \in D_\varepsilon$, of the singular point z is $\mathcal{S}$-compatible if, for any $t_1, t_2 \in D_\varepsilon \backslash \{0\}$, there exist bijections $\mathrm{Sing}(V_{t_1}) \leftrightarrow \mathrm{Sing}(V_{t_2})$ and $\mathrm{Sing}(W_{t_1}) \leftrightarrow \mathrm{Sing}(W_{t_2})$ such that the corresponding points in $\mathrm{Sing}(V_{t_1})\backslash L$ and $\mathrm{Sing}(V_{t_2})\backslash L$ are $\mathcal{S}$-equivalent as usual singular points, the corresponding points in $\mathrm{Sing}(W_{t_1})$ and $\mathrm{Sing}(W_{t_2})$ are $\mathcal{S}$-equivalent as boundary singularities of V_{t_1}, V_{t_2} with respect to L.

Let $D \subset \mathbb{C}^n$ be homeomorphic to a closed $2n$-ball, and $Y \subset D$ be a topological $(2n-2)$-manifold with boundary $\partial Y = Y \cap \partial D$. Assume that $\mathrm{Cl}(Y)\backslash Y^*$ is a finite set in $\mathrm{Int}(D)$, and in a neighborhood of any point $z \in \mathrm{Cl}(Y)\backslash Y$, the set $\mathrm{Cl}(Y)$ is a complex $(n-1)$-manifold with isolated singularity at z.

In the above notation, the pair (D, Y) is said to be a model for an $\mathcal{S}$-compatible deformation F_t, $t \in D_\varepsilon$, of the singular point z of F if, for any $t \neq 0$, there exists a homeomorphism $(B, \{F_t = 0\}) \to (B(R_0), Y)$ which takes $L \cap B$ to $L \cap D$, $\mathrm{Sing}(F_t)$ to $\mathrm{Sing}(Y)$ and $\mathrm{Sing}(F_t|_L)$ to $\mathrm{Sing}(Y \cap L)$ so that the corresponding singular points in $\mathrm{Sing}(F_t)\backslash L$ and $\mathrm{Sing}(Y)\backslash L$ are $\mathcal{S}$-equivalent as usual singularities, the corresponding points in $\mathrm{Sing}(F_t|_L)$ and $\mathrm{Sing}(Y \cap L)$ are $\mathcal{S}$-equivalent as L-boundary singularities of (F_t, Y).

Next we formulate a criterion for the existence of one-parametric deformations of NNDT singular points with given models. An important requirement for models will be $\mathcal{S}$-transversality, which we define later in accordance with the similar notion in [21], section 3.

2.3.2 BUV deformations of an NNDT singular point with given models. Let z be an NNDT singular point of multiplicity r of an algebraic hypersurface $F \subset \mathbb{C}P^n$ of degree d. Let F^* be the proper transform of F under the blowing-up $\pi: \Sigma \to \mathbb{C}P^n$ of the point z, E be the exceptional divisor, $z_1, \ldots, z_k$ be all the singular points of $F^* \cap E$. Any point z_s, $1 \leq s \leq k$, is Newton nondegenerate with respect to suitable local coordinates $x_1, \ldots, x_n$ in which $z_s = (0, \ldots, 0)$, $E = \{x_n = 0\}$, and F^* is given by a polynomial equation $F_s(x_1, \ldots, x_n) = 0$.

---

* $\mathrm{Cl}(Y)$ denotes the closure in the metric topology.

Here we assume that the truncations of F_s of the facets of the Newton diagram $\Gamma(F_s)$ are nondegenerate. Introduce the ideals

$$I_s = I_{z_s|E} + \left\{ \sum_{\mathbf{i}\in\Delta(F_s)\backslash\Gamma(F_s)} A_{\mathbf{i}}\mathbf{x}^{\mathbf{i}} \right\} \subset \mathcal{O}_{\Sigma,z_s}$$

and the zero-dimensional schemes X_s defined by them for all $s = 1, \ldots, k$.

THEOREM 4: *In the above notation, let $(D^{(i)}, Y^{(i)})$, $i = 1, \ldots, k$, be $\mathcal{S}$-transversal models for $\mathcal{S}$-compatible deformations of the points $z_1, \ldots, z_k$, respectively.*

(i) *If $n = 2$ then there exists a one-parametric fixed equimultiple deformation F_t, $F_0 = F$, $\deg F_t = d$, of the singular point z of F such that the family of proper transforms F_t^* realizes an $\mathcal{S}$-compatible deformation of the points $z_1, \ldots, z_k$ with the given models $(D^{(i)}, Y^{(i)})$, $i = 1, \ldots, k$.*

(ii) *If $n > 2$ and*

$$H^1(E, \mathcal{J}_{(X_1\cup\ldots\cup X_k)\cap E/E} \otimes \mathcal{O}_E(r)) = 0, \tag{10}$$

then there exists a one-parametric fixed equimultiple deformation F_t, $F_0 = F$, $\deg F_t = d$, of the singular point z of F such that the family of proper transforms F_t^* realizes an $\mathcal{S}$-compatible deformation of the points $z_1, \ldots, z_k$ with the given models $(D^{(i)}, Y^{(i)})$, $i = 1, \ldots, k$.

Remark 2: If $n = 2$, then any model (D, Y) which has ordinary nodes as its only singularities is $\mathcal{S}$-transversal by Corollary 2 [21].

2.3.3 $\mathcal{S}$-transversal models for deformations of semiquasihomogeneous boundary singular points. In the above notation, let the origin in $\mathbb{C}^n$ be a semiquasihomogeneous L-boundary singular point with the main face σ of the Newton diagram of F, i.e., the truncations F^σ and $F^\sigma\big|_L$ (the sums of monomials in F, $F\big|_L$ corresponding to the integral points in σ) have no critical points in $\mathbb{C}^n\backslash\{0\}$, $L\backslash\{0\}$, respectively. Introduce the space of polynomials

$$\mathcal{P}(\sigma) = \left\{ \sum_{l(\mathbf{i})\geq 0} \alpha_{\mathbf{i}}\mathbf{x}^{\mathbf{i}} \right\},$$

where $l(i_1, \ldots, i_n)$ is a linear function, vanishing on σ and positive at the origin.

Let a polynomial $P \in \mathcal{P}(\sigma)$ with $P^\sigma = F^\sigma$ define a hypersurface $W = \{P = 0\} \subset \mathbb{C}^n$ having only isolated singular points. There is $R_0 > 0$ such that any sphere S_R of radius $R \leq R_0$ centered at the origin intersects W transversally. Denote by $D \subset \mathbb{C}^n$ the closed ball of radius R_0 centered at the origin, and consider the pair (D, Y), where $Y = W \cap B(R_0)$, as a model for an $\mathcal{S}$-compatible deformation of the singular point z of F.

Given an integer $l_0 > \max\{i_1 + \cdots + i_n \colon l(i_1, \ldots, i_n) \geq 0\}$, we consider P, $\mathcal{P}(\sigma)$, $\mathcal{P}(\sigma, F)$ to be included in the space $\mathcal{P}(l_0)$ of polynomials of degree $\leq l_0$. For any singular point w of $W \cap B^{2n}$, introduce the germ $M_d(w, P)$ at P of the set of polynomials in $\mathcal{P}(d)$, the space of polynomials of degree d, which are close to P and have a singular point $\mathcal{S}$-equivalent to (W, w) in a neighborhood of w. We say that a model (D, Y) for deformations of the semiquasihomogeneous singular point $z \in W$ satisfies the $\mathcal{S}$-transversality (resp., strong $\mathcal{S}$-transversality) condition if there exists a sufficiently large integer d such that

- the intersection $M_d(P)$ of the germs $M_d(w, P)$, $w \in \mathrm{Sing}(W \cap B^{2n})$, is smooth, and
- $M_d(P)$ intersects with $\mathcal{P}(\sigma)$ (resp., $\mathcal{P}(\sigma, F)$) in $\mathcal{P}(d)$ transversally.

2.3.4 $\mathcal{S}$-transversal models for deformations of Newton nondegenerate boundary singular points. Now let the origin in $\mathbb{C}^n$ be a Newton nondegenerate L-boundary singular point with the Newton diagram $\Gamma(F)$ of a polynomial of F, i.e., the truncations F^σ and $F^\sigma\big|_L$ (the sums of monomials in F, $F\big|_L$ corresponding to the integral points in σ) to the facets σ of $\Gamma(F)$ have no critical points in $\mathbb{C}^n \backslash \{0\}$, $L \backslash \{0\}$, respectively.

Denote by $K(\Gamma(F))$ the bounded closed domain in the nonnegative orthant $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \geq 0\}$ bounded by $\Gamma(F)$ and the coordinate hyperplanes. Let us be given

- a subdivision of $K(\Gamma(F))$ into convex lattice polyhedra $\Delta_1, \ldots, \Delta_N$ and a convex piece-wise linear real-valued function ν on $\mathbb{R}^n_+$ such that $\Delta_1, \ldots, \Delta_N, \Delta(F)$ are its linearity domains, $\nu\big|_{\Delta(F)} = 0$,
- a set of numbers $\{A_{\mathbf{i}} \colon \mathbf{i} \in (K(\Gamma(F)) \backslash \Gamma(F)) \cap \mathbb{Z}^n_+\}$ such that, for any Δ_k and any proper face $\sigma \subset \Delta_k$, the polynomial $F^\sigma(\mathbf{x}) = \sum_{\mathbf{i} \in \sigma} A_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ has no singular points in $(\mathbb{C}^*)^n$.

Here we assume that the coefficients $A_{\mathbf{i}}$, $\mathbf{i} \in \Gamma(F)$, are those of the polynomial F, and that all the polynomials $F_k(\mathbf{x}) = \sum_{\mathbf{i} \in \Delta_k} A_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$ and $F_k\big|_L$ define hypersurfaces in $(\mathbb{C}^*)^n$, respectively in $(\mathbb{C}^*)^{n-1} \subset L$, with isolated singular points only.

The following construction has been introduced in [15]. For any $k = 1, \ldots, N$ we have the moment map of the positive orthant $\mathbb{R}^n_+$ into the polyhedron Δ_k

$$\mu_k(\mathbf{x}) = \frac{\sum_{\mathbf{i} \in \Delta_k} \mathbf{i} \cdot \mathbf{x}^{\mathbf{i}}}{\sum_{\mathbf{i} \in \Delta_k} \mathbf{x}^{\mathbf{i}}} \stackrel{\text{def}}{=} (\mu_k^{(1)}(\mathbf{x}), \ldots, \mu_k^{(n)}(\mathbf{x})),$$

which is a diffeomorphism of $\mathbb{R}^n_+$ onto $\mathrm{Int}(\Delta_k)$ (see [2, 3]). We introduce the complexification of the polyhedron Δ_k

$$\mathbb{C}\Delta_k = \{(w_1, \dots, w_n) \in \mathbb{C}^n \colon (|w_1|, \dots, |w_n|) \in \Delta_k\},$$

and the complexification of the moment map

$$\mathbb{C}\mu_k \colon (\mathbb{C}^*)^n \to \mathbb{C}\Delta_k, \quad \mathbb{C}\mu_k(w_1, \dots, w_n) = (\mu_k^{(1)}(\mathbf{x})v_1, \dots, \mu_k^{(n)}(\mathbf{x})v_n),$$
$$\mathbf{x} = (|w_1|, \dots, |w_n|), \quad v_1 = \frac{w_1}{|w_1|}, \dots, v_n = \frac{w_n}{|w_n|},$$

which is a diffeomorphism of $(\mathbb{C}^*)^n$ onto $\mathrm{Int}(\mathbb{C}\Delta_k) \cap (\mathbb{C}^*)^n$. By the (complex) chart $\mathrm{Ch}(F_k)$ of the polynomial F_k we call the closure of the set $\mathbb{C}\mu_k(\{F_k = 0\} \cap (\mathbb{C}^*)^n)$.

PROPOSITION 4: (i) *The set* $\mathbb{C}K(\Gamma(F)) = \mathbb{C}\Delta_1 \cup \dots \cup \mathbb{C}\Delta_N$ *is homeomorphic to a closed ball* B^{2n}. *The set* $W = \mathrm{Ch}(F_1) \cup \dots \cup \mathrm{Ch}(F_N)$ *is a (topological)* $(2n-2)$*-manifold with the boundary* $W \cap \partial(\mathbb{C}K(\Gamma(F)))$ *and the finite singular locus* $\mathrm{Sing}(W) = \mathbb{C}\mu_1(\mathrm{Sing}(F_1)) \cup \dots \cup \mathbb{C}\mu_N(\mathrm{Sing}(F_N))$.

(ii) *Let* $\{F = 0\}$ *intersect the boundary of a closed ball* B^{2n} *centered at the origin transversally, and* $\{F = 0\} \cap B^{2n}$ *be a cone over* $\{F = 0\} \cap \partial(B^{2n})$. *Let functions* $\widetilde{A}_{\mathbf{i}}(t)$, $t \geq 0$, *with* $\widetilde{A}_{\mathbf{i}}(0) = A_{\mathbf{i}}$, $\mathbf{i} \in \mathbb{Z}^n_+$, *be such that, for a sufficiently small* $t > 0$, *there is a bijection between the singular point set of the hypersurface* $\widetilde{F}(\mathbf{x}) = \sum \widetilde{A}_{\mathbf{i}}(t)\mathbf{x}^{\mathbf{i}}t^{\nu(\mathbf{i})} = 0$ *in* $\mathrm{Int}(B^{2n})$ *and the disjoint union* $\mathrm{Sing}(F_1, \dots, F_N)$ *of the sets* $\mathrm{Sing}(F_1) \cap (\mathbb{C}^*)^n, \dots, \mathrm{Sing}(F_N) \cap (\mathbb{C}^*)^n$, *so that the corresponding singular points are* $\mathcal{S}$*-equivalent. Then there exists a homeomorphism of* $(B^{2n}, \{\widetilde{F} = 0\} \cap B^{2n})$ *onto* $(\mathbb{C}K(\Gamma(F)), W)$ *which extends the above bijection* $\mathrm{Sing}(\widetilde{F}) \cap B^{2n} \leftrightarrow \mathrm{Sing}(F_1, \dots, F_N)$.

If all the data are defined over the reals and $\mathrm{Sing}(F_1, \dots, F_N) = \emptyset$, the statement of Proposition 4 for the real parts of $\mathbb{C}K(\Gamma(F))$, W, B^{2n} and $\{\widetilde{F} = 0\}$ is a particular case of Viro's theorem [24]. In fact, the proof of Viro's theorem [24] provides the same claim with $\mathrm{Sing}(F_1, \dots, F_N) \neq \emptyset$. In full generality Proposition 4 is proven in [15] in a similar way.

We consider the pair $(D, Y) = (\mathbb{C}K(\Gamma(F)), W)$ as a model for an $\mathcal{S}$-compatible deformation of the singular point z of F.

We shall define the $\mathcal{S}$-transversality for the above model (D, Y) in accordance with [21], section 3.

Let G be the dual graph of the subdivision $\mathbb{R}^n_+ = \Delta(F) \cup \Delta_1 \cup \dots \cup \Delta_N$: namely, its vertices are incident to $\Delta(F), \Delta_1, \dots, \Delta_N$, and its arcs are incident to common facets of these polyhedra. Define $\mathcal{G}$ to be the set of oriented graphs $\widetilde{G}$ with support G, without oriented cycles, and such that no arc goes out of the vertex corresponding to $\Delta(F)$. It is clear that $\mathcal{G} \neq \emptyset$. For any $\widetilde{G} \in \mathcal{G}$ we denote by $\Delta_{i,+}(\widetilde{G})$ the union of facets of Δ_i, which correspond to the arcs of $\widetilde{G}$ coming in Δ_i, $i = 1, \dots, N$.

For a polynomial $F_i(\mathbf{x})$ and $l \geq \deg F_i$ denote by $M_l(F_i)$ the germ at F_i of the set of polynomials of degree $\leq l$ which, in a neighborhood of $\mathrm{Sing}(F_i) \cap (\mathbb{C}^*)^n$, define singular points $\mathcal{S}$-equivalent to the corresponding singular points of $\{F_i = 0\}$, and in a neighborhood of $\mathrm{Sing}(F_i|_L) \cap (\mathbb{C}^*)^{n-1} \subset L$ define L-boundary singular points $\mathcal{S}$-equivalent to the corresponding singular points of $\{F_i = 0\}$. The triple $(\Delta_i, \Delta_{i,+}(\widetilde{G}), F_i)$ is called $\mathcal{S}$-transversal if

- $M_l(F_i)$ is smooth for sufficiently large l,
- if $n \geq 3$, then $M_l(F_i)$ $(l >> 0)$ intersects transversally with the set of polynomials with Newton polyhedron Δ_i having the same coefficients at $\mathbf{i} \in \Delta_{i,+}(\widetilde{G})$ as F_i; if $n = 2$, then $M_l(F_i)$ $(l >> 0)$ intersects transversally with the set of polynomials with Newton polyhedron Δ_i, whose truncation on any connected component of $\Delta_{i,+}(\widetilde{G})$ is proportional to the corresponding truncation of F_i.

The model $(\mathbb{C}K(\Gamma(F)), W)$ is called $\mathcal{S}$-transversal if there exists a graph $\widetilde{G} \in \mathcal{G}$ such that all the triples $(\Delta_i, \Delta_{i,+}(\widetilde{G}), F_i)$, $i = 1, \dots, N$, are $\mathcal{S}$-transversal.

Remark 3: In contradiction to the case $n = 2$, for $n \geq 3$, we do not know reasonable general $\mathcal{S}$-transversality criteria, even for nodal models (cf. Remark 2), and any particular case requires a special consideration. We illustrate this in the following example.

Example 3: In the above notation, suppose that, for $i = 1, \dots, N$,

- the hypersurface $\{F_i = 0\}$ is nonsingular in $(\mathbb{C}^*)^n$, if the set $\Delta_i \backslash \Delta_{i,+}(\widetilde{G})$ contains no integral points,
- the hypersurface $\{F_i = 0\}$ has a node as its only singularity in $(\mathbb{C}^*)^n$, if the set $\Delta_i \backslash \Delta_{i,+}(\widetilde{G})$ contains an integral point.

Then the corresponding model $(\mathbb{C}K(\Gamma(F)), W)$ is $\mathcal{S}$-transversal. For, one has to show that any triple $(\Delta_i, \Delta_{i,+}(\widetilde{G}), F_i)$ with $\mathbb{Z}^n \cap \Delta_i \backslash \Delta_{i,+}(\widetilde{G}) \neq \emptyset$ is $\mathcal{S}$-transversal. Let $z \in (\mathbb{C}^*)^n$ be the node of $\{F_i = 0\}$. Then the tangent space at F_i to $M_l(F_i)$ consists of polynomials vanishing at z. On the other hand, the space $\mathcal{P}(\Delta_i, \Delta_{i,+}(\widetilde{G}), F_i)$ contains the family $F_i + \lambda \mathbf{x}^{\mathbf{i}}$, $\lambda \in \mathbb{C}$, where $\mathbf{i}$ is an integral point in $\Delta_i \backslash \Delta_{i,+}(\widetilde{G})$, and this family is transverse to $T_{F_i}(M_l(F_i))$, because $\mathbf{x}^{\mathbf{i}}$ does not vanish at z.

2.3.5 Proof of Theorem 4. As in the proof of Theorem 1 [21], the existence of the required one-parametric deformation F_t^* follows from the $\mathcal{S}$-transversality of the models $(B^{(i)}, Y^{(i)})$, $i = 1, \ldots, k$, and the condition

$$H^1(\Sigma, \mathcal{J}_{X_1 \cup \cdots \cup X_k/\Sigma} \otimes \mathcal{O}_\Sigma(d\widetilde{L} - rE)) = 0, \tag{11}$$

where $\widetilde{L} \subset \Sigma$ is the preimage of a generic hyperplane in $\mathbb{C}P^n$ (note that $F^* \in |d\widetilde{L} - rE|$). For any $s = 1, \ldots, k$, the ideal I_s defining X_s contains the ideal $\{\sum_{l_s(\mathbf{i})<0} A_\mathbf{i}\mathbf{x}^\mathbf{i}\}$, which in turn contains the germ of $(d - r + 1)E$. Then we can complete the proof along the argument in the proof of Theorem 3, provided (1) holds for $n \geq 2$. The latter ought to be verified only for $n = 2$. Indeed, in this case the scheme X_s, $1 \leq s \leq k$, on the line E is defined by the ideal $(\mathfrak{m}_{z_s})^{m_s - 1} \subset \mathcal{O}_{E,z_s}$, where m_s is the intersection multiplicity of E and F^*, which immediately implies (1), since $m_1 + \cdots + m_k \leq r$.

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